About one inequality from APMO, 2004 (New solution and generalizations).

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Original problem from **APMO, 2004/5** is:
Prove that

$$\left(a^2+2\right)\left(b^2+2\right)\left(c^2+2\right) \geq 9\left(ab+bc+ca\right)^2$$

for any positive real numbers a, b, c.

1. We will prove more stronger inequality

(1) $\left(a^2+2\right)\left(b^2+2\right)\left(c^2+2\right) \geq 3\left(a+b+c\right)^2$

which yields original inequality (because $(a+b+c)^2 \geq 3(ab+bc+ca)$).
Since for any real x, y holds inequalities

(2) $\left(x^2+2\right)\left(y^2+2\right) \geq 2\left(x+y\right)^2 \iff (xy-2)^2 \geq 0$ and

(3) $\left(x^2+2\right)\left(y^2+2\right) \geq \dfrac{3}{2}\left((x+y)^2+2\right) \iff 2\left(xy-1\right)^2+\left(x-y\right)^2 \geq 0$

then

$$\left(a^2+2\right)\left(b^2+2\right)\left(c^2+2\right) \geq \frac{3}{2}\left((a+b)^2+2\right)\left(c^2+2\right) \geq \frac{3}{2}\cdot 2\left((a+b)+c\right)^2 = 3\left(a+b+c\right)^2$$

with equality condition $a=b=c$, which follows immediately from equality condition for **(2)** and **(3)**.

2. Generalization 1.

For any positive real numbers x, y, z, t holds inequality

(4) $\left(x^2+t^2\right)\left(y^2+t^2\right)\left(z^2+t^2\right) \geq \dfrac{3t^4}{4}\left(x+y+z\right)^2.$

Proof.

Since for any real x, y and t holds inequalities

(5)

$$\left(x^2+t^2\right)\left(y^2+t^2\right) \geq t^2\left(x+y\right)^2 \iff \left(xy-t^2\right)^2 \geq 0$$

and

(6)

$$\left(x^2+t^2\right)\left(y^2+t^2\right) \geq \frac{3t^2}{4}\left((x+y)^2+t^2\right) \iff \left(xy-\frac{t^2}{2}\right)^2+\frac{t^2}{4}\left(x-y\right)^2 \geq 0$$

then

$$\left(x^2+t^2\right)\left(y^2+t^2\right)\left(z^2+t^2\right)\geq\frac{3t^2}{4}\left((x+y)^2+t^2\right)\left(z^2+t^2\right)\geq\frac{3t^2}{4}\cdot t^2\left((x+y)+z\right)^2=\frac{3t^4}{4}\left(x+y+z\right)^2$$

with equality condition $x=y=z=\dfrac{t}{\sqrt{2}}$, which follows immediately from equality condition for **(5)** and **(6)**.
Those inequalities can be obtained from the similar inequalities in the particular case $a=\sqrt{2}$ by substitution $a=\dfrac{\sqrt{2}x}{t}$,$b=\dfrac{\sqrt{2}y}{t}$, $c=\dfrac{\sqrt{2}z}{t}$.

3. Generalization 2.
Let $x,y,t>0$ then

(7) $\quad\left(x^n+t^n\right)\left(y^n+t^n\right)\geq\dfrac{t^n}{2^{n-2}}\left(x+y\right)^n, n\geq 2.$

Indeed, since by Power Mean Inequality

$$\left(\frac{y^n+t^n}{2}\right)^{\frac{1}{n}}\geq\left(\frac{y^{\frac{n}{n-1}}+t^{\frac{n}{n-1}}}{2}\right)^{\frac{n-1}{n}}\iff$$

$$\frac{y^n+t^n}{2}\geq\left(\frac{y^{\frac{n}{n-1}}+t^{\frac{n}{n-1}}}{2}\right)^{n-1}\iff$$

$$y^n+t^n\geq\frac{\left(y^{\frac{n}{n-1}}+t^{\frac{n}{n-1}}\right)^{n-1}}{2^{n-2}}$$ and by Holder's Inequality

$$\left(x^n+t^n\right)\left(y^{\frac{n}{n-1}}+t^{\frac{n}{n-1}}\right)^{n-1}=\left(x^n+t^n\right)\left(t^{\frac{n}{n-1}}+y^{\frac{n}{n-1}}\right)^{n-1}\geq\left(xt+ty\right)^n=t^n\left(x+y\right)^n$$

then $\left(x^n+t^n\right)\left(y^n+t^n\right)\geq\dfrac{1}{2^{n-2}}\left(x^n+t^n\right)\left(y^{\frac{n}{n-1}}+t^{\frac{n}{n-1}}\right)^{n-1}\geq\dfrac{t^n}{2^{n-2}}\left(x+y\right)^n.$

4. Generalization 3.
For any natural m and n such that $2\leq m\leq n$ and positive real numbers $x_1,x_2,...,x_m,t$ holds inequality

(8) $\quad\displaystyle\prod_{k=1}^{m}\left(x_k^n+t^n\right)\geq\frac{t^{n(m-1)}}{m^{n-m}\left(m-1\right)^{m-1}}\cdot\left(x_1+x_2+...+x_m\right)^n$

with equality condition $x_k=\dfrac{t}{\sqrt[n]{m-1}}, k=1,2,...,m.$

Proof.
Denote $P:=\displaystyle\prod_{k=1}^{m}\left(x_k^n+t^n\right)$ and $S:=\displaystyle\sum_{k=1}^{m}\frac{1}{x_k^n+t^n}$ then by AM-GM Inequality we have

$$\frac{x_i^n}{x_i^n+t^n}+\sum_{k=1,k\neq i}^{m}\frac{t^n}{x_k^n+t^n}\cdot\frac{1}{m-1}=\frac{x_i^n}{x_i^n+t^n}-\frac{t^n}{(m-1)\left(x_i^n+t^n\right)}+\frac{St^n}{m-1}\geq$$
$$m\sqrt[m]{\frac{x_i^nt^{n(m-1)}}{P\left(m-1\right)^{m-1}}}, i=1,2,...,m.$$

Summing this inequalities we obtain:
$\sum_{i=1}^{m} \frac{x_i^n}{x_i^n+t^n} + \frac{mt^nS}{m-1} - \frac{t^n}{m-1}\sum_{i=1}^{m}\frac{1}{x_i^n+t^n} = \sum_{i=1}^{m}\frac{x_i^n}{x_i^n+t^n} + t^nS =$
$\sum_{i=1}^{m} \frac{x_i^n}{x_i^n+t^n} + \sum_{k=1}^{m}\frac{t^n}{x_k^n+t^n} = \sum_{i=1}^{m}\frac{x_i^n+t^n}{x_i^n+t^n} = m \geq$

$$m\sum_{i=1}^{m}\sqrt[m]{\frac{x_i^n t^{n(m-1)}}{P\left(m-1\right)^{m-1}}} \iff P^{\frac{1}{m}} \geq \frac{t^{\frac{n(m-1)}{m}}\cdot\sum_{i=1}^{m}x^{\frac{n}{m}}}{\left(m-1\right)^{\frac{m-1}{m}}} \iff$$

$$P \geq \frac{t^{n(m-1)}}{\left(m-1\right)^{m-1}}\left(\sum_{i=1}^{m}x^{\frac{n}{m}}\right)^m = \frac{t^{n(m-1)}m^m}{\left(m-1\right)^{m-1}}\left(\frac{\sum_{i=1}^{m}x^{\frac{n}{m}}}{m}\right)^m.$$

Since by Power Mean Inequality

$$\left(\frac{\sum_{i=1}^{m}x^{\frac{n}{m}}}{m}\right)^{\frac{m}{n}} \geq \frac{\sum_{i=1}^{m}x_i}{m} \iff \left(\sum_{i=1}^{m}x^{\frac{n}{m}}\right)^m \geq \frac{\left(\sum_{i=1}^{m}x_i\right)^n}{m^{n-m}}$$

then we finally get inequality

$$P \geq \frac{t^{n(m-1)}}{\left(m-1\right)^{m-1}m^{n-m}}\left(\sum_{i=1}^{m}x_i\right)^n,$$

where equality occurs iff $x_1 = x_2 = ... = x_m = \frac{t}{\sqrt[n]{m-1}}$
(condition of equality in Power Mean Inequality).

Applications.
Problem 3326(a).(CRUX,Vol.34,No.3)
Let a, b,and c be positive real numbers.
(a) Show that $\prod_{cyc}\left(a^2+2\right) + 4\prod_{cyc}\left(a^2+1\right) \geq 6\left(a+b+c\right)^2$.
Solution.
Applying inequality **(4)** to $t=\sqrt{2}$ and $t=1$ we obtain
$\prod_{cyc}\left(a^2+2\right) \geq \frac{3\cdot\left(\sqrt{2}\right)^4}{4}\left(a+b+c\right)^2 = 3\left(a+b+c\right)^2$ and
$\prod_{cyc}\left(a^2+1\right) \geq \frac{3\cdot 1^4}{4}\left(a+b+c\right)^2 = \frac{3}{4}\left(a+b+c\right)^2$.
Hence, $\prod_{cyc}\left(a^2+2\right) + 4\prod_{cyc}\left(a^2+1\right) \geq 6\left(a+b+c\right)^2$.
Problem 3327(a).(CRUX,Vol.34,No.3)
Let a, b,and c be positive real numbers.
(a) Show that $\prod_{cyc}\left(a^4+3a^2+2\right) \geq \frac{9}{4}\left(a+b+c\right)^2$.
Solution.
$\prod_{cyc}\left(a^4+3a^2+2\right) = \prod_{cyc}\left(a^2+2\right)\cdot\prod_{cyc}\left(a^2+1\right) \geq 3\left(a+b+c\right)^2\cdot\frac{3}{4}\left(a+b+c\right)^2 =$

$\frac{9}{4}(a+b+c)^2$.

Remark.
There are many solutions of this APMO **2004** problem 5, but all what I sow are more complicate and heavy.
For example: 3 solutions of Problem 67 in "Old and New Inequalities" Titu Andreescu and Vasile Cirtoaje, and 2 solutions in "Topics in Inequalities" of Hojoo Lee and also solution in
https://mks.mff.cuni.cz/kalva/apmo/asoln/asol045.html

